Section 12.5: Lines and Planes

A line in 3-space is given by parameterized vector equation

$$\ell(t) = \vec{p} + t\vec{v}$$

where $\vec{p}$ = position vector of a point on ℓ, and $\vec{v}$ = direction of line

Ex: Compute the vector equation of the line through $(-6,2,3)$ and parallel to line $m(t) = \langle 0,2,-1\rangle + t\langle -2,1,5\rangle$

Sol: Given $\vec{p} = \langle -6,2,3\rangle$. because of parallelism $\vec{v} = \langle -2,1,5\rangle$ is a valid direction vector for the desired line

$\therefore \ell(t) = \langle -6,2,3\rangle + t\langle -2,1,5\rangle$

The parametric equations of a line are

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases} \leftarrow \text{Component functions of the vector form}$$

Ex: For ℓ and m as in the previous example, we simplify vector equation $\ell(t) = \langle -6-2t,\ 2+t,\ 3+5t\rangle$

$\hookrightarrow$ $\therefore$ has parametric equations $\begin{cases} x = -6-2t \\ y = 2+t \\ z = 3+5t \end{cases}$

$m(t) = \langle -2t,\ 2+t,\ -1+5t\rangle$

$\hookrightarrow$ $\therefore$ has parametric equations $\begin{cases} x = -2t \\ y = 2+t \\ z = -1+5t \end{cases}$

A line can also be represented (sometimes) by symmetric equation.

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \leftarrow \text{(solved for parameter)}$$

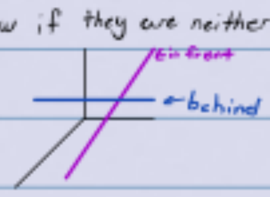
Ex: For ℓ as above we had parametric equations:

$$\begin{cases} x = -6-2t \\ y = 2+t \\ z = 3+5t \end{cases} \rightsquigarrow \begin{cases} \frac{x+6}{-2} = t \\ \frac{y-2}{1} = t \\ \frac{z-3}{5} = t \end{cases} \rightsquigarrow \frac{x+6}{-2} = \frac{y-2}{1} = \frac{z-3}{5}$$

are the symmetric equations of ℓ

Some Terminology: Two lines are...

1. parallel if their direction vectors are parallel
2. Intersecting if they have a common point
3. Skew if they are neither parallel nor intersecting



Ex: Classify as parallel, intersecting, or skew:

$\ell_1(t): \langle 5-12t,\ 3+9t,\ 1-3t\rangle$

$\ell_2(t): \langle 3+8t,\ -6t,\ 7+2t\rangle$

Sol: $\ell_1(t) = \overbrace{\langle 5,3,1\rangle}^{\vec{p}_1} + t\overbrace{\langle -12,9,-3\rangle}^{\vec{v}_1}$ $\quad \ell_2(t) = \overbrace{\langle 3,0,7\rangle}^{\vec{p}_2} + t\overbrace{\langle 8,-6,2\rangle}^{\vec{v}_2}$

Unit Vectors: $\frac{1}{|\vec{v}_1|}\vec{v}_1 = \frac{1}{\sqrt{234}}\langle -12,9,-3\rangle = \frac{1}{3\sqrt{26}}\langle -12,9,-3\rangle = \frac{1}{\sqrt{26}}\langle -4,3,-1\rangle$

$\frac{1}{|\vec{v}_2|}\vec{v}_2 = \frac{1}{\sqrt{104}}\langle 8,-6,2\rangle = \frac{1}{2\sqrt{26}}\langle 8,-6,2\rangle = \frac{1}{\sqrt{26}}\langle 4,-3,1\rangle$ ← minus 1 scalar multiple

$\frac{1}{|\vec{v}_1|}\vec{v}_1 = -\frac{1}{|\vec{v}_2|}\vec{v}_2$, so ℓ_1 is parallel to ℓ_2

Sol (Intersect):

Check if ℓ_1 and ℓ_2 intersect:

Not looking for collison of line $\ell_1(t_0) = \ell_2(t_0)$ rather that they cross the same point at any (even if different) times

Solve $\ell_1(t) = \ell_2(s)$ i.e. $\langle 5-12t,\ 3+9t,\ 1-3t\rangle = \langle 3+8s,\ -6s,\ 7+2s\rangle$

$$\begin{cases} 5-12t = 3+8s \\ 3+9t = -6s \\ 1-3t = 7+2s \end{cases} \stackrel{\Leftrightarrow}{=} \begin{cases} -12t+8s = -2 \\ 9t+6s = -3 \\ -3t-2s = 6 \end{cases} \rightsquigarrow \begin{cases} 6t-4s = 1 \\ 3t+2s = -1 \\ 3t+2s = -6 \end{cases}$$

can't both be true

implies $-1 = 3t+2s = 6$ $\quad -1 \neq 6$!

So these lines do not intersect

Recall: A plane in 3-space has vector equation $\vec{n}\cdot(\vec{x}-\vec{p})=0$

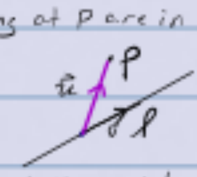
normal vector ($\vec{n}$), vector of variables ($\vec{x}$), position any point on the plane ($\vec{p}$)

Ex: Compute the plane through $(1,2,4)$ and perpendicular to $\langle -2,1,3\rangle$

Sol: $\vec{n}\cdot(\vec{x}-\vec{p})=0 = \langle -2,1,3\rangle\cdot\langle x-1, y-2, z-4\rangle = 0$ $\quad -2(x-1)+1(y-2)+3(z-4)=0$

Any vector starting on line and ending at P are in plane

Ex (2): Compute the plane through the point $(3,5,-1)$ and containing the line $\begin{cases} x=4-t \\ y=2t-1 \\ z=-3t \end{cases}$



both must be in the plane

Sol: $\vec{P}=\langle 3,5,-1\rangle$ need Q, a point on ℓ. let's use $Q=\ell(0)=(4,-1,0)$

$\vec{u}=\langle 3-4, 5-^-1, -1-0\rangle = \langle -1,6,-1\rangle$ $\quad \vec{v}$ = a direction vector of the line

$\ell(t)=\langle 4,-1,0\rangle + t\underbrace{\langle -1,2,-3\rangle}_{\vec{v}}$ direction vector from parametric form

$\vec{u}\times\vec{v}=\vec{n} = \begin{vmatrix} i & j & k \\ -1 & 6 & -1 \\ -1 & 2 & -3 \end{vmatrix} = i\begin{vmatrix} 6 & -1 \\ 2 & -3 \end{vmatrix} - j\begin{vmatrix} -1 & -1 \\ -1 & -3 \end{vmatrix} + k\begin{vmatrix} -1 & 6 \\ -1 & 2 \end{vmatrix} = i(18-^-2)-j(3-1)+k(-2-^-6)$

$\vec{n}=\langle -16,-2,4\rangle$

$\vec{n}\cdot(\vec{x}-\vec{p})=0 = \langle -16,-2,4\rangle\cdot\langle x-3, y-5, z+1\rangle = 0$ $\quad -16(x-3)-2(y+5)+4(z+1)=0$

Section 12.6: Quadratic Surfaces: (textbook calls it Quadric surfaces)

IDEA: We want to study degree 2 polynomials and their solution sets in 3-space. ← hard in general...

Ex: $P(x,y,z)=x^2-z$ ← "degenerate" because it doesn't depend on all of the variables (could just be done in x,z plane)

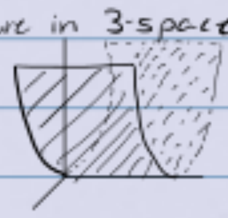
Solution set: $p(x,y,z)=0$ if and only if $x^2+z=0$ if and only if $x^2=z$

Picture in xz-plane:

$y=0$ $\quad$ This solution set is actually a (parabolic) cylinder:

one "slice" at any singular y value

Picture in 3-space



"kind of like a skateboard ramp" because parabola solution same for all values of y

folding a paper

It turns out, "up to" translation, reflection and rotation, there are only 6 nondegenerate quadratic surfaces...

Name	Equation
Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
Cone	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$
Elliptic Paraboloid	
	Save for next time